Lesson 68  Triangle Congruency; Triangle Proofs

Review: Lessons 9, 10, 66 and 67.

Rules

- Triangle congruency theorems*

**Side-angle-side (SAS):** If two sides and the included angle in one triangle have the same measures as two sides and the included angle in a second triangle, the triangles are congruent. It is based off of Euclid’s proposition 4.

**Side-side-side (SSS):** If the lengths of the sides in one triangle are equal to the lengths of the sides in a second triangle, the triangles are congruent. It is based off Euclid’s Proposition 8.

**Angle-angle-angle-side (AAAS):** If the angles in one triangle equal the angles in a second triangle, the triangles are similar. If it is also known that at least one pair of sides opposite the same angle measure are congruent, then the two triangles are also congruent. Also known as ASA (because if two angles are congruent, then so are the third angles), it is based off of Euclid’s Proposition 26.

**Hypotenuse-Leg (HL):** If the lengths of the hypotenuse and a leg in one right triangle equal the lengths of the hypotenuse and a leg in a second right triangle, the right triangles are congruent.

*You may also see some of the previous four theorems described as postulates elsewhere. Because Euclid’s propositions describe most of these, we will refer to them as theorems, which is another word for proposition.*

**Third angle theorem:** If two angles in one triangle are congruent to two angles in another triangle, the third angles are also congruent.

Definitions

**CPCTC** - Acronym for **C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent

≡ - Symbol used for the phrase “is congruent to.”

**angle bisector** - a ray that divides an angle into two congruent angles.
68A Triangle Congruency

As you learned in Lesson 66, the concept of proof, and proof technique, are not just ideas to remind you that you are doing geometry homework! Proof technique can be applied in an unending variety of real-world applications. Completing proofs is often challenging, and requires an extra level of patience, perseverance and self control. They force you to think things through a little more before you commit to saying or writing something for the rest of the world to see. Supporting statements with reasons is a technique used by, and expected of, people that society refers to with words like professional, leader, wise, helpful, and trustworthy. People like Abraham Lincoln, the 16th President of the United States of America, known for his study of Euclid’s Elements and his application of the idea of proof to solving societal problems.

Lessons like this one are common in high school geometry courses, and are designed to give you practice with writing proofs. First, we will get familiar with the 4 triangle congruency theorems. Then we will apply them to complete some triangle congruency proofs.

**Example 68.1** Match each pair of congruent triangles with the congruency theorem that best describes it.

![Triangle Congruency Diagrams](image)

**solution:** In congruency relationships, “tick marks” are used to identify congruent sides and angles. Note that some sides and angles have one, two, or three tick marks. These are clues to help you identify which congruency theorem matches which triangle pair.

- A) SSS, because three congruent sides are identified.
- B) HL, because the hypotenuse and one leg are congruent.
- C) SAS, because two sides and their included angle are congruent.
- D) AAAS, because three angles and one side are congruent.
68B Triangle Proofs

Let’s apply the triangle congruency theorems now to create some two-column proofs. Before completing the proof, a good first step is to use “tick marks” to identify the given information on the diagram, plus any other information that is self-evident. Doing this can help you determine what triangle congruency theorem to use in your proof.

Example 68.2

Given $AD \cong DB$

$DC \perp AB$

Prove $\triangle ACD \cong \triangle BCD$

solution: First, let’s use tick marks to identify the given information. Also, given the fact that $DC$ is perpendicular to $AB$ is a clue that we have two right triangles. Also, since side $CD$ is shared by both triangles, then $CD \cong CD$, which is Euclid’s Axiom 4 (a.k.a reflexive axiom). A circle is normally used to identify Axiom 4.

With the diagram labeled, it is clear that the two triangles have 2 congruent sides and one congruent angle. Therefore, the triangles are congruent by SAS. Do you see how sketching the given information on the diagram, plus Axiom 4, helped us with the proof?

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) $AD \cong DB$</td>
<td>Given</td>
</tr>
<tr>
<td>2) $DC \perp AB$</td>
<td>Given</td>
</tr>
<tr>
<td>3) $\angle ADC = \angle BDC = 90^\circ$</td>
<td>Lines or line segments forming $90^\circ$ angles are perpendicular to each other (Lesson 9).</td>
</tr>
<tr>
<td>4) $CD \cong CD$</td>
<td>Axiom 4</td>
</tr>
<tr>
<td>5) $\triangle ACD \cong \triangle BCD$</td>
<td>SAS</td>
</tr>
</tbody>
</table>
Example 68.3  Given $\overline{AC} \cong \overline{BC}$  
$\overline{DC} \perp \overline{AB}$  
Prove $\triangle ACD \cong \triangle BCD$

**solution:** This is similar to Example 68.2, except that we were given different information about the sides. We have to write the proof based on the information given in this problem, not the previous one, so ignore the previous problem. Sketch the problem, and notice you are comparing two right triangles, with enough information to show that the hypotenuse and one leg are congruent. Therefore, the two triangles are congruent by HL.

**Statements**

1) $\overline{AC} \cong \overline{BC}$ 
2) $\overline{DC} \perp \overline{AB}$ 
3) $\angle ADC = \angle BDC = 90^\circ$

4) $\overline{CD} \cong \overline{CD}$
5) $\triangle ACD \cong \triangle BCD$

**Reasons**

Given

Given

Lines or line segments forming $90^\circ$ angles are perpendicular to each other (Lesson 9).

Axiom 4

HL

Example 68.4  Given $\overline{AC}$ bisects $\angle BAD$ and $\angle BCD$  
Prove $\overline{AB} \cong \overline{AD}$

**solution:** If we can prove triangles $ABC$ and $ADC$ are congruent, then we can prove sides $AB$ and $AD$ are congruent by CPCTC. By our definition of an angle bisector, we know angles $BAC$ and $DAC$, and $BCA$ and $DCA$ are congruent. The third angle theorem reveals that angles $ABC$ and $ADC$ are congruent. Since both triangles have side $AC$ in common, we can prove the two triangles are congruent by AAAS. But we’re not finished! We still need to say that sides $AB$ and $AD$ are congruent by CPCTC.
<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( \overline{AC} ) bisects ( \angle BAD ) and ( \angle BCD )</td>
<td>Given</td>
</tr>
<tr>
<td>2) ( \angle BAC \cong \angle DAC, \angle BCA \cong \angle DCA )</td>
<td>Angle bisector definition</td>
</tr>
<tr>
<td>3) ( \angle ABC \cong \angle ADC )</td>
<td>Third angle theorem</td>
</tr>
<tr>
<td>4) ( \overline{AC} \cong \overline{AC} )</td>
<td>Axiom 4</td>
</tr>
<tr>
<td>5) ( \triangle ABC \cong \triangle ADC )</td>
<td>AAAS</td>
</tr>
<tr>
<td>6) ( \overline{AB} \cong \overline{AD} )</td>
<td>CPCTC</td>
</tr>
</tbody>
</table>

**Practice Set 68** Use your best judgement as to when you should and shouldn’t use a calculator. Use 3.14 for \( \pi \).

1. **68**. Match each pair of congruent triangles with the congruency theorem that best describes it.

   A) [Diagram A]
   
   B) [Diagram B]
   
   C) [Diagram C]
   
   D) [Diagram D]

2. **68**. Given \( \overline{AB} \cong \overline{AD} \)

   \[ \overline{BC} \cong \overline{DC} \]

   Prove \( \triangle ABC \cong \triangle ADC \)

3. **66**. Given a circle of radius 5 cm, prove that its area equals \( 25 \pi \) cm\(^2\). Use the formula for the area of a circle (Lesson 18) in your proof.

4. **65**. Evaluate. \( \lim_{x \to 0} \frac{3x^3 + 2x^2}{x^2} \)

5. **64**. Solve the system of equations shown. Write the answer as an ordered pair.

   \[ \begin{cases} 
   y + x = 7 \\
   y = 3x - 5 
   \end{cases} \]

6. **63**. A bag contained 5 green marbles, 12 yellow marbles, and 2 blue marbles. What is the probability of a person removing a yellow marble?
7. Simplify. \( \frac{1 + 1}{h} + \frac{1}{5} \cdot x \)

8. Which of the following best represents the local minimum visible in the graph of the function \( f(x) = x^3 + 3x^2 + 1 \)?
   A) (-2, 5)  
   B) (-1,3)  
   C) (0,1)  
   D) (-1,0)

9. (Chemistry) Density \( D \) equals mass \( M \) divided by volume \( V \). In other words, \( D = \frac{M}{V} \).

   Find the volume in mL of 300 g of acetone at 25 °C if its density at this temperature is 0.79 g/mL. Round answer to 1 decimal place.

10. Identify the function represented by the graph.
   A) \( f(x) = 3x + 1 \)  
   B) \( f(x) = |x| + 1 \)
   C) \( f(x) = 2x^2 + 1 \)  
   D) \( f(x) = 2x^3 + 2x^2 + 1 \)

11. Factor. \( x^2 + 4x - 5 \)

12. Find the measure of arc EFG.

13. Simplify. \( \sqrt[4]{x^3} \cdot x^2 \)

14. The graduated cylinders cost $12.10 each for 1 to 3, $11.07 each for 4-47, and $10.20 each for 48 or more. What is the maximum number of graduated cylinders the scientist can purchase for $500?
15. (CLEP College Math) If a and b represent prime numbers, what is the greatest common divisor of $a^5b^3$ and $a^3b^5$? Hint: the greatest common divisor is the same thing as the greatest common factor.

A) $a^3b^3$  
B) $a^5b^4$  
C) $a^2b^{-1}$  
D) $a^8b^7$

16. The speed of sound in air decreases with altitude. The table below lists standard values for the speed of sound in air from 0 to 10 km. Find the mean speed of sound from 0 to 10 km, rounded to 1 decimal place.

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>Speed of Sound (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>340.3</td>
</tr>
<tr>
<td>1</td>
<td>336.4</td>
</tr>
<tr>
<td>2</td>
<td>332.5</td>
</tr>
<tr>
<td>3</td>
<td>328.6</td>
</tr>
<tr>
<td>4</td>
<td>324.6</td>
</tr>
<tr>
<td>5</td>
<td>320.5</td>
</tr>
<tr>
<td>6</td>
<td>316.5</td>
</tr>
<tr>
<td>7</td>
<td>312.3</td>
</tr>
<tr>
<td>8</td>
<td>308.1</td>
</tr>
<tr>
<td>9</td>
<td>303.8</td>
</tr>
<tr>
<td>10</td>
<td>299.5</td>
</tr>
</tbody>
</table>

17. Convert 100 °F to Celsius. Round answer to 1 decimal place.

18. Who painted the famous one point perspective painting, *The Last Supper*?  
A) DaVinci  
B) Picasso  
C) Gauss  
D) Rembrandt

19. Factor. $a^5b^3 + a^3b^4$

20. Add algebraically. $+5 + (-9) + 20 - 11 + (+1)$